香港城市大學
City University of Hong Kong

396EM Airline Operations and Scheduling/ 6075MAA Airline Scheduling and Operations

Lecture 1b
Linear Programming (Review)

Developed & Revised :
Teaching team, 204KM (for Introduction to Linear Programming)
Kimman Lui & Rossella Lau (Scope)
Presented: Kinki Leung

專業 創新 胸懷全球
Professional·Creative
For The World

SCOPE
School of Continuing and Professional Education
Coventry University

Linear Programming

- Linear programming, mathematical modeling technique useful for guiding quantitative decisions in business planning, industrial engineering, and—to a lesser extent—in the social and physical sciences.

- During World War II, linear programming was used extensively to deal with transportation, scheduling, and allocation of resources subject to certain restrictions such as costs and availability.

SOLVE THIS …

- A company wishes to schedule the production of a kitchen appliance which requires 2 resources: labour & materials.
- The company is considering two different models (A and B) & its production engineering department has provided the following data.
- Supply of raw materials is restricted to £240 per day.
- The daily availability of man power is 320 hours.
- Formulate a linear programming model to determine the daily production rate of the various models in order to maximize the total profit.

	A	B
Material ($$/Unit)	1	1
Labour (Man/Unit)	2	1
Profit (£ / Unit)	**40**	**30**

Linear Programming (LP)

- The previous example illustrates a typical linear programming problem.
- LP is too versatile to be completely described by any single example.
- So how did you formulate the problem?

Formulation of LP Problem

- Step 1:
  - The unknown activities to be determined are the daily rate of production for the 2 models.
  - How are they to be represented?
  - Represent them by algebraic symbols
    - X_A
    - X_B
- Step 2 …

Formulation of LP Problem (Con't)

- Step 2:
  - Express them as linear equations or inequalities which are linear functions of the unknown variables.
  - In this example the constraints are the limited availability of the 2 resources

Labour Constraint

Material Constraint

  - In addition restrict the variables $X_{A,}$ X_B to have only non-negative values.
  - This is known as the non-negativity constraints.

Non-negativity Constraint

- Step 3 …

Formulation of LP Problem (Con't)

- Step 3: Identify the objective & represent it as a linear function of the decision variables, which is to be maximized or minimized

  - In this example the objective is to maximize the total profits from sales.
  - Assuming that a perfect market exists for the product such that all that is produced can be sold.

  - Objective Function:

    Maximize $40X_A + 30X_B$

- Now Solve It!

Formulation of LP Problem (Con't)

- The linear programming (LP) model for the example becomes:
- Maximize $Z = 40X_A + 30X_B$
- Subject to the constraints

$X_A + X_B \leq 240$	Materials Constraint
$2X_A + X_B \leq 320$	Labour Constraint
$X_A, X_B \geq 0$	Non-negativity Constraint

Solving it!

- Now you need to solve it!
- Which means what?
- Which means in this example you need to find values of X_A and X_B which maximizes the objective function, but does not violate any of the constraints.
- Now have a go at solving this LP model!

Common Terminology

EXAMPLE	GENERAL PROBLEM * [General terms for the same components that will fit most linear programming models]
Limited Resource available 2 resources i.e. labour & raw materials	Resources m resources
Production of products 2 products i.e. Models A and B Production rate of product j (x_j)	Activities n activities Level of activity j (x_j)
Profit (Z)	Overall performance measure (Z)

Key Terms

- Resources
  - Need to perform activities
  - The number of each is denoted by ‘m’.
  - The amount available of each resource is limited. E.g. time/materials etc
  - Hence careful allocation of resources to activities must be made.
- Activities
  - The number of each is denoted by ‘n’.

Standard Form of LP Model

- Can now formulate the mathematical model for this general problem of allocating resources to activities

- For this model have to select values for $X_1, X_2, X_3, \ldots X_n$ which are the **decision variables**.

So as to Maximize $\qquad Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$

Subject to the following constraints

$$
\begin{aligned}
a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n &\le b_1 \\
a_{21}X_1 + a_{22}X_2 + \ldots + a_{2n}X_n &\le b_2 \\
&\ldots \\
&\ldots \\
a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n &\le b_m
\end{aligned}
$$

And $X_1 \ge 0, X_2 \ge 0, \ldots, X_n \ge 0$

Standard Form of LP Model (Con't)

- Any situation whose mathematical formulation fits this model is a linear programming problem.

- In our example:
  - $m = 2$, $n = 2$, $b_1 = 240$ & $b_2 = 320$

- Write them down.

Other Forms of LP Model

There are other legitimate forms of LP Models such as:

- Minimizing rather than Maximizing the objective function
  - So as to Minimize $Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n$
- Some functional constraints with a greater than or equal to inequality
  - $a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{in}X_n \geq b_i$ for some value of i
- Some functional constraints in equation form
  - $a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{in}X_n = b_i$ for some value of i
- Deleting the non-negativity constraint for some decision variables.

Terminology For The LP Solution

- Solution – Any specification of values for the decision variables (X1, X2, . . . , Xn). Regardless of whether it is a desirable or even an allowable choice.
- Feasible Solution – A solution for which all the constraints are satisfied. Note it is possible to have no feasible solution.
- Feasible Region – Collection of all feasible solutions.
- Given that there are many feasible solutions, the goal of linear programming is to find which one is best as measured by the value of the objective function in the model.

Terminology For The LP Solution(Con't)

- Optimal Solution – This is a feasible solution that has the most favourable value of the objective function.

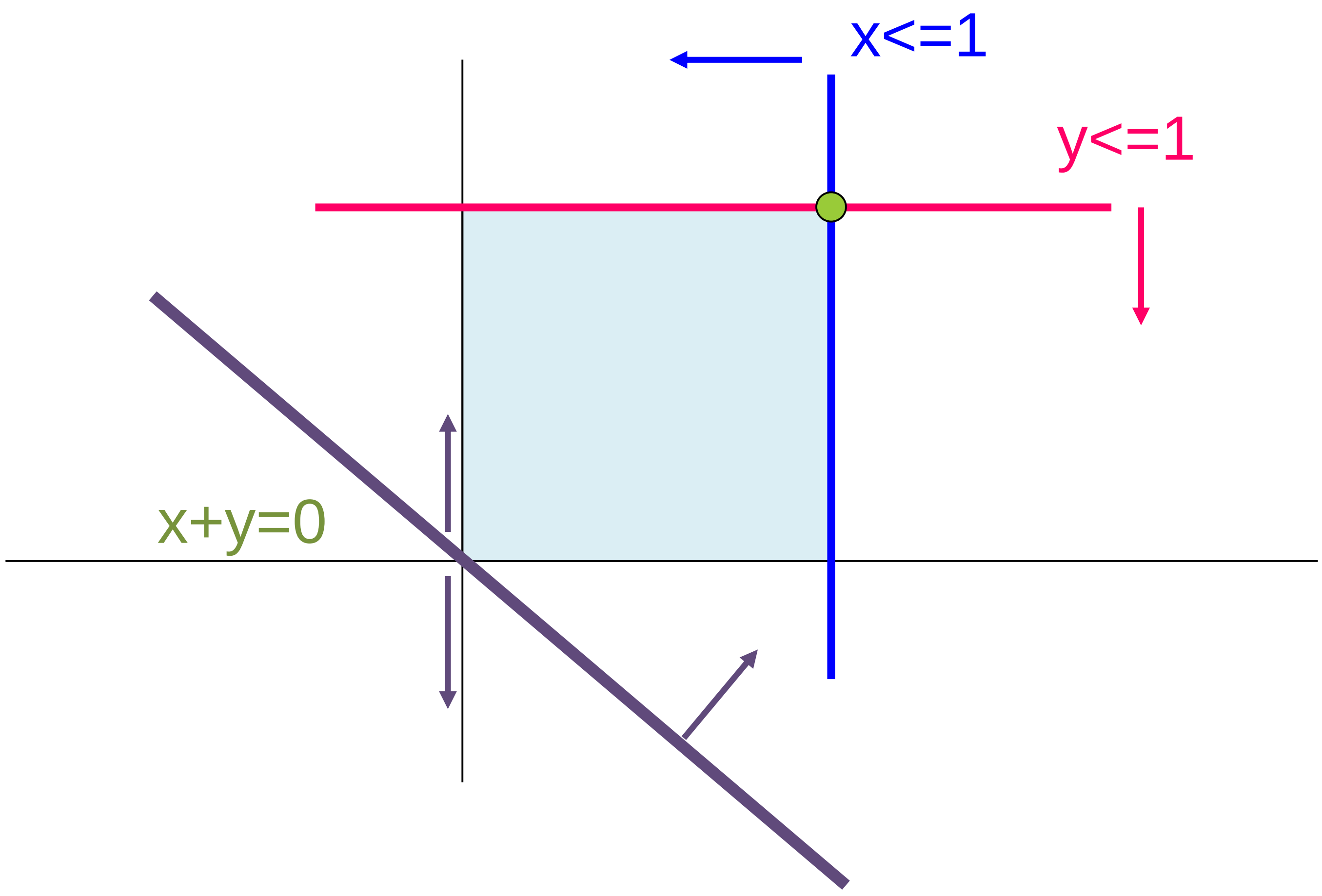
- Most problems have just one optimal solution. However it is possible to have more than one or even none !

- Most favourable means the largest or the smallest value depending upon whether the objective is maximisation or minimisation.

Solving LP Problem

- Maximize $40x+30y$ → maximize $x+y$
- $x>=0$
- $y>=0$
- $x+y <= 240$ → $x <= 1$
- $2x+y <= 320$ → $y <= 1$

Solving LP Problem (Con't)



Solution

Constraints

$x \geq 0$

$y \geq 0$

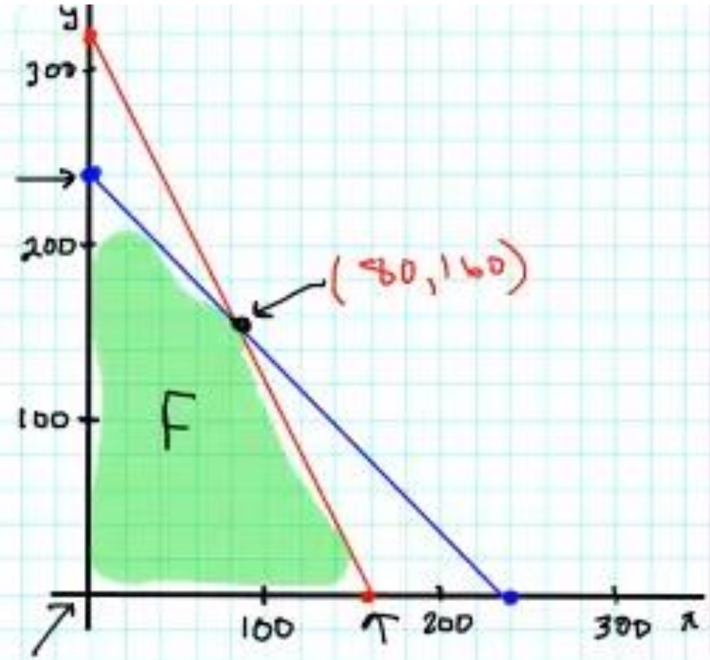
$x + y \leq 240$

$2x + y \leq 320$

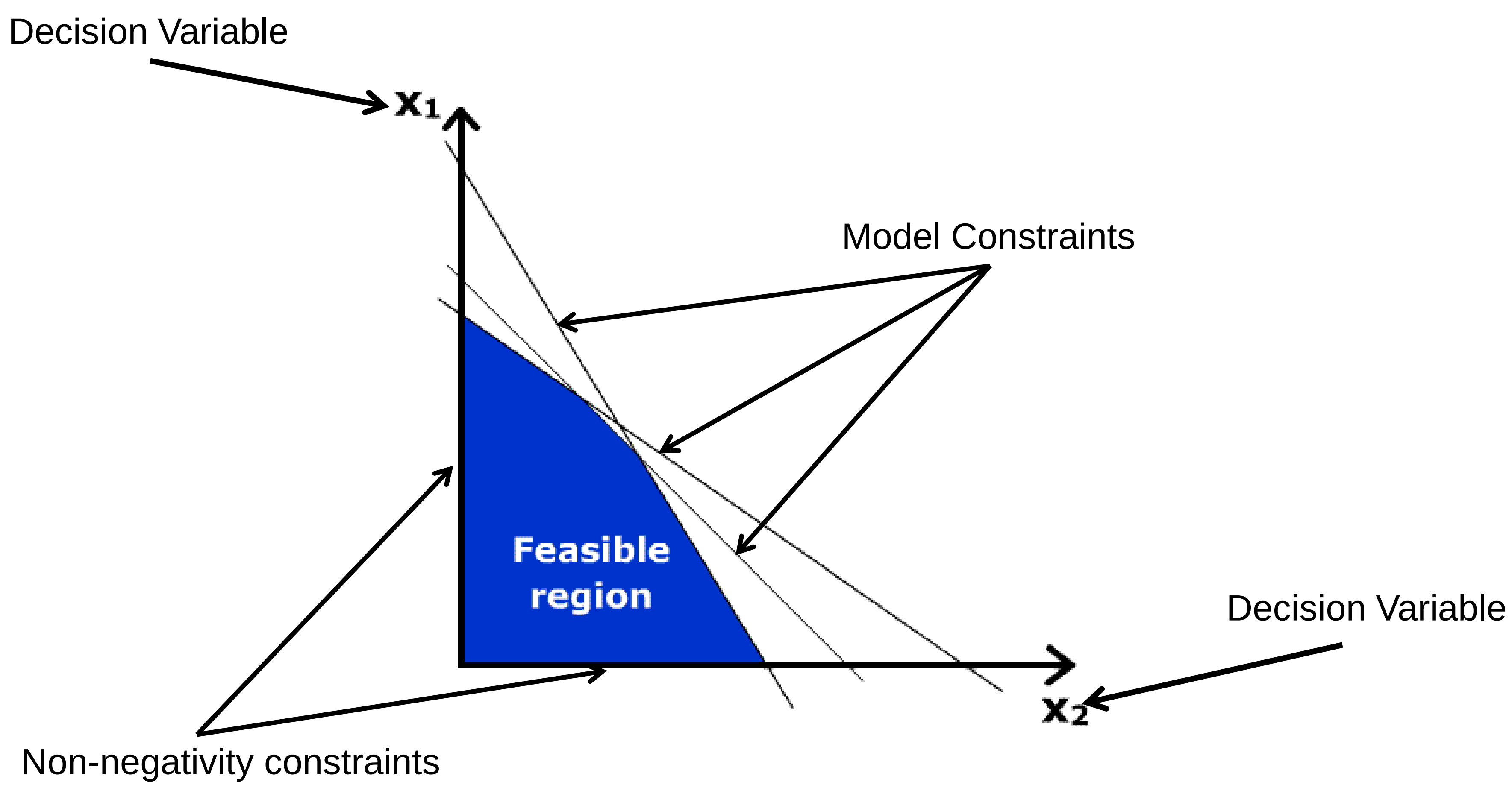
$$-x - \cancel{y} = -240$$
$$2x + \cancel{y} = 320$$
$$x = 80$$

$$80 + y = 240$$
$$-80 \quad\quad -80$$
$$y = 160$$

x	y	$P = 40x + 30y$
0	0	0
160	0	6400
0	240	7200
80	160	3200 + 4800 = 8000

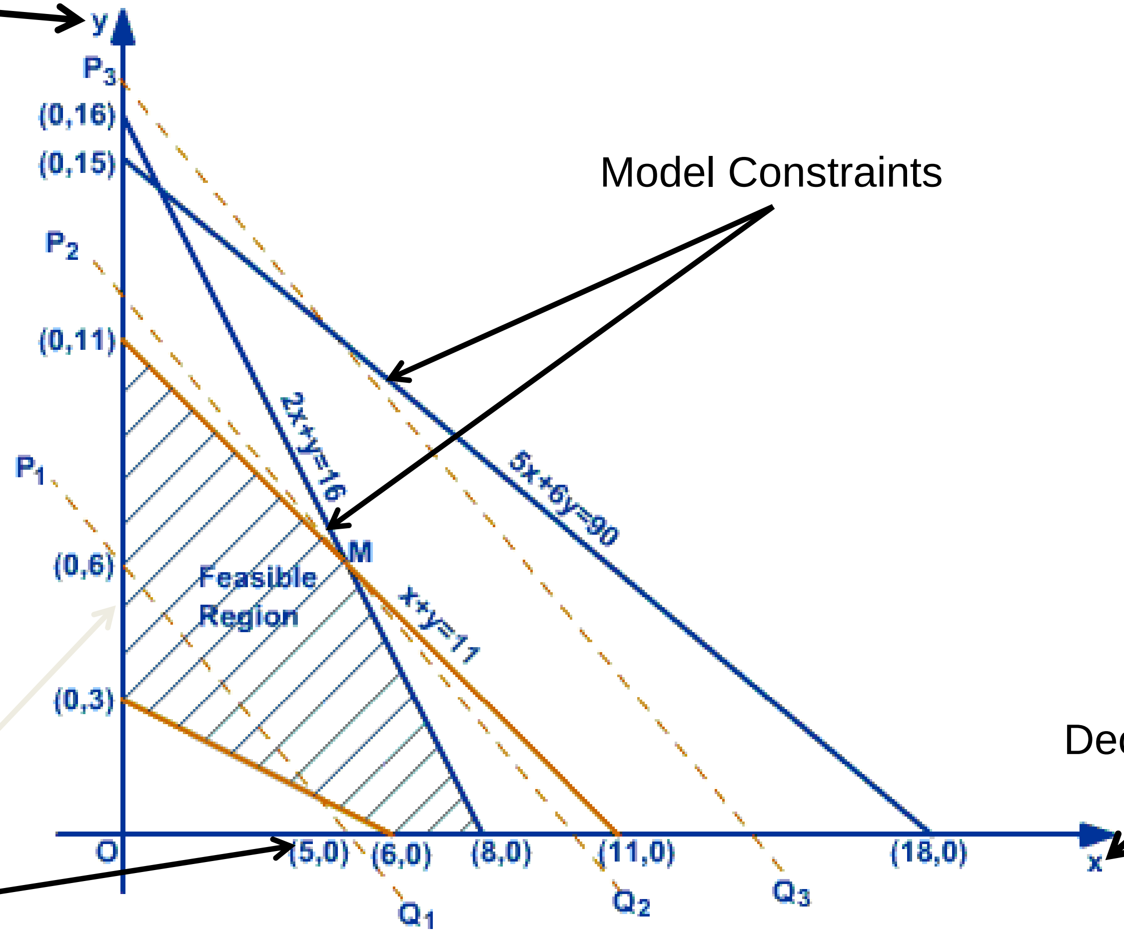


Feasible Region In 2D



Feasible Region In 2D (Con't)

Decision Variable



Model Constraints

Decision Variable

Non-negativity constraints

Feasible Region In 2D (Con't)

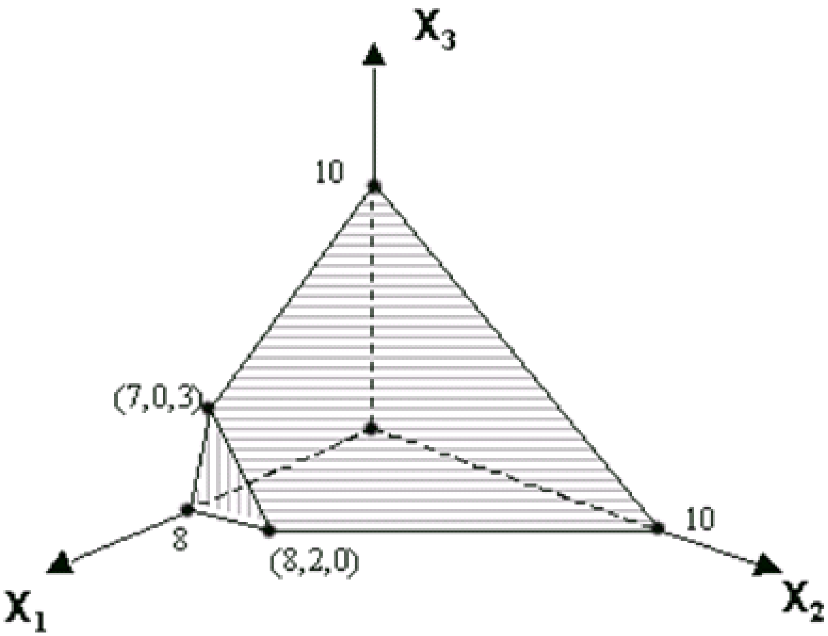
- The feasible region is constructed with just straight lines.
  - Hence the word **'linear'** in linear programming.
  - Each straight line represents a constraint from the model.
  - Together the constraints enclose and define the feasible region.

- With 2 decision variables you can plot the feasible region on a graph.

- What about when there are 3 decision variables?

- Where does the **optimum solution lie**?
  - In the feasible region: In the middle, on the edge, on the corner?

Feasible Region In 3D



- Look at the straight edges.
- They define the boundaries of the feasible region.
- The solution of this LP problem must lie in the feasible region.
- But where does the optimal solution lie?

Recommend Readings

- Operation Research
http://people.brunel.ac.uk/~mastjjb/jeb/or/intro.html
http://people.brunel.ac.uk/~mastjjb/jeb/or/basicor.html

- Applications of OR in the Air Transport Industry
https://pubsonline.informs.org/doi/pdf/10.1287/trsc.37.4.368.23276

- Integer Linear Programming
https://www.doc.ic.ac.uk/~br/berc/integerprog.pdf