香港城市大學
City University of Hong Kong

396EM Airline Operations and Scheduling / 6075MAA Airline Scheduling and Operations

Lecture 1c
Airline Network Flows and Integer Programming Models

Developed & Revised :
Dr Yang Dai & Dr Ammar Bazi (Coventry)
Kimman Lui & Rossella Lau (Scope)
Presented: Kinki Leung

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Integer Linear Programming

- A large part of the problems that airlines face can be translated into network and integer programming models.
- In short, an integer programming problem is a mathematical optimization or feasibility program in which some or all of the variables are restricted to be integers
- We provide a review of some of the optimization models

Network Flow models

1. Shortest Path (Router) problem

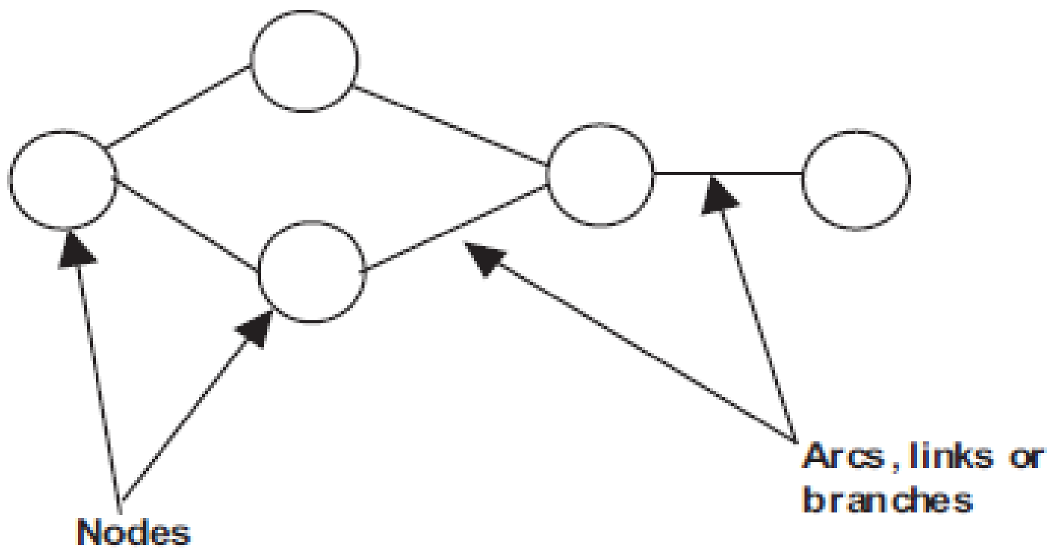
2. Minimum cost flow problem

3. Maximum flow problem

4. Multi-commodity Problem

Network

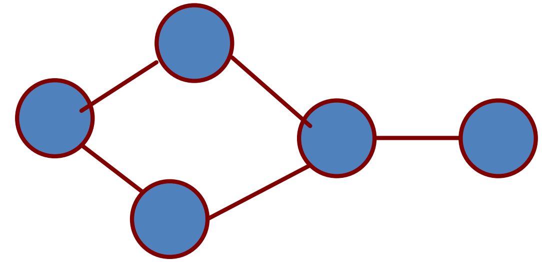
- A network (also referred to as a graph) is defined as a collection of points and lines joining these points.



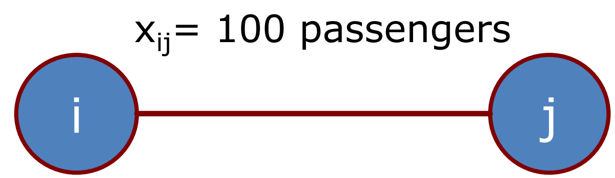
Basic elements of a network

Network Terminology

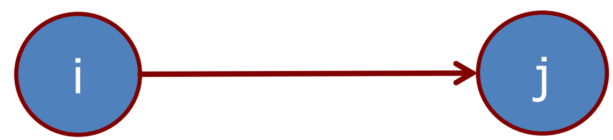
- Nodes and Arcs: in a network, the points (circles) are called nodes and the lines are referred to as arcs, links, or arrows.



- Flow: the amount of goods, flights, passengers, and etc. that move from one node to another.



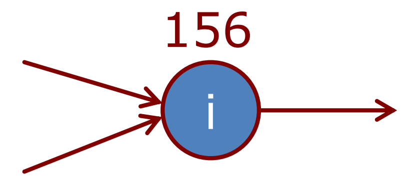
- Directed Arc: if the flow through an arc is allowed only in one direction, then the arc is said to be a directed arc.



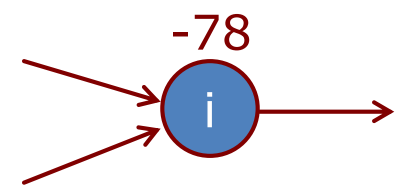
- Undirected Arc: When the flow on an arc can be move in either direction, it is called an undirected arc.

Supply Node and Demand Node

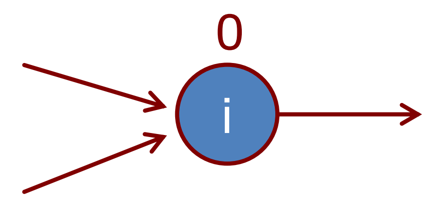
- Arc Capacity: the maximum amount of flow that can be sent through an arc, e.g. restrictions on the flight no. between 2 cities.



- Supply nodes: Nodes with the amount of flow coming to them greater than the amount of flow leaving them, or nodes with positive net flow.

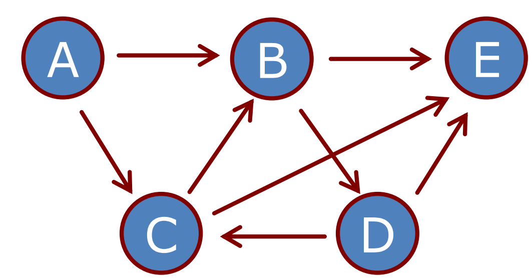


- Demand Nodes: Nodes with negative net flow or outflow greater than inflow.



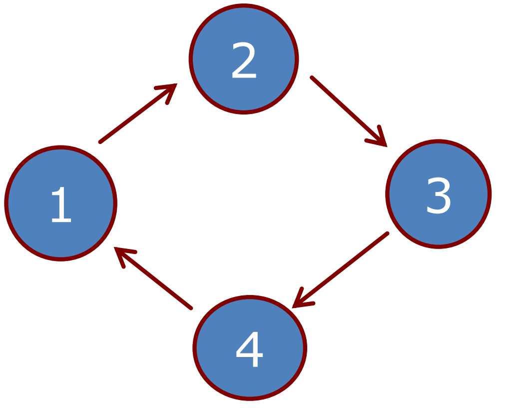
- Transshipment: Nodes with the same amount of flow arriving and leaving or nodes with zero net flow.

- Path: a path is a sequence of distinct arcs that connect 2 nodes in this fashion since sometimes 2 nodes are not connected by an arc, but could be connected by a sequence of arcs. E.g. Airlines utilize hubs to provide connections between city pairs in their network.

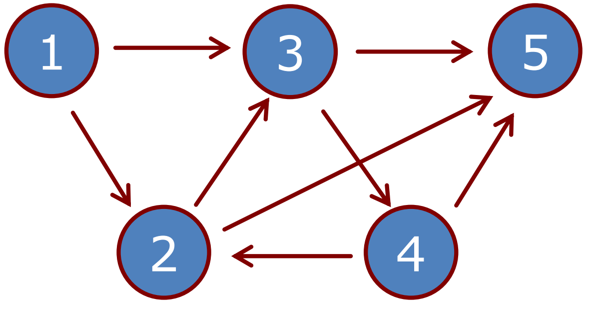


Cycle and Connected Network

- Source: Starting node in the path.
- Destination: Last node in the path.
- Cycle: A sequence of directed arcs that begins and ends at the same node. E.g. Aircraft start from an airport which is a maintenance base and, after flying to several destinations, end up at the same airport from which they departed.



- Connected Network: A network in which every two nodes are linked by at least one path.



Network Flow models

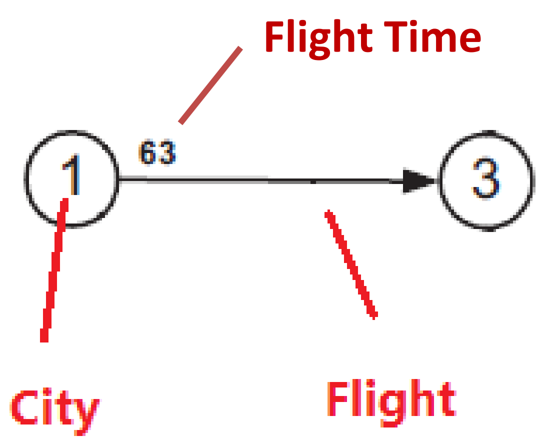
1. Shortest Path (Router) problem
2. Minimum cost flow problem
3. Maximum flow problem
4. Multi-commodity Problem

1. Shortest Path (Route) problem

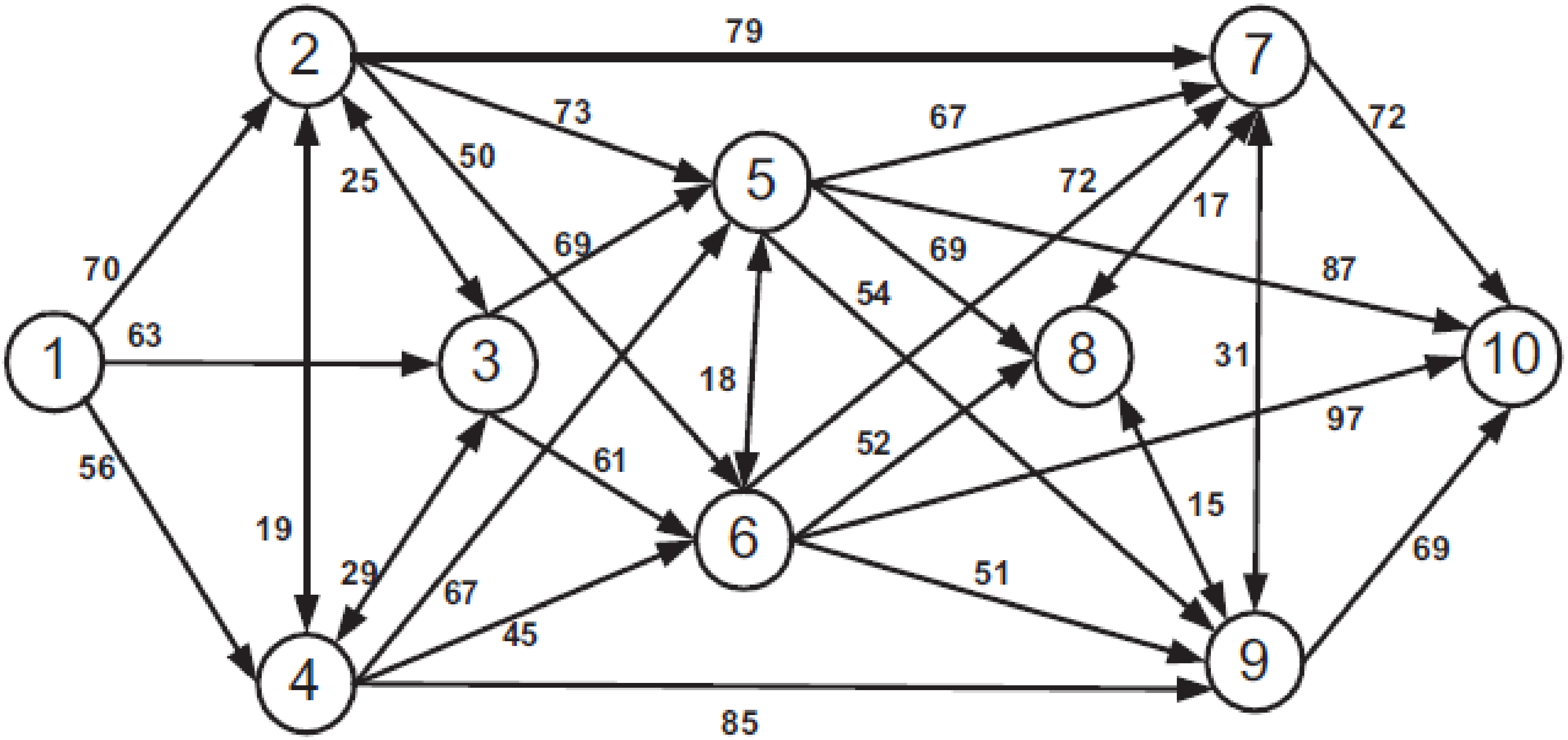
- Attempts to identify a path, from source to destination, within the network, that results in minimum transport time/cost.
- The objective is to identify the path with the minimum cost between two nodes.
- The problem consists of a connected network with known costs for each arc in the network
- Typical application for air cargo handlers, freight forwarders and etc. to identify the shortest route to transport cargos through indirect flights

Notation:
Network with flight times between city pairs

- The nodes represent the cities, and the arcs are the flights.
- The numbers on the arcs represent the flight time in minutes between the city pairs.
- We want to determine the best route that results in the shortest flying time from node

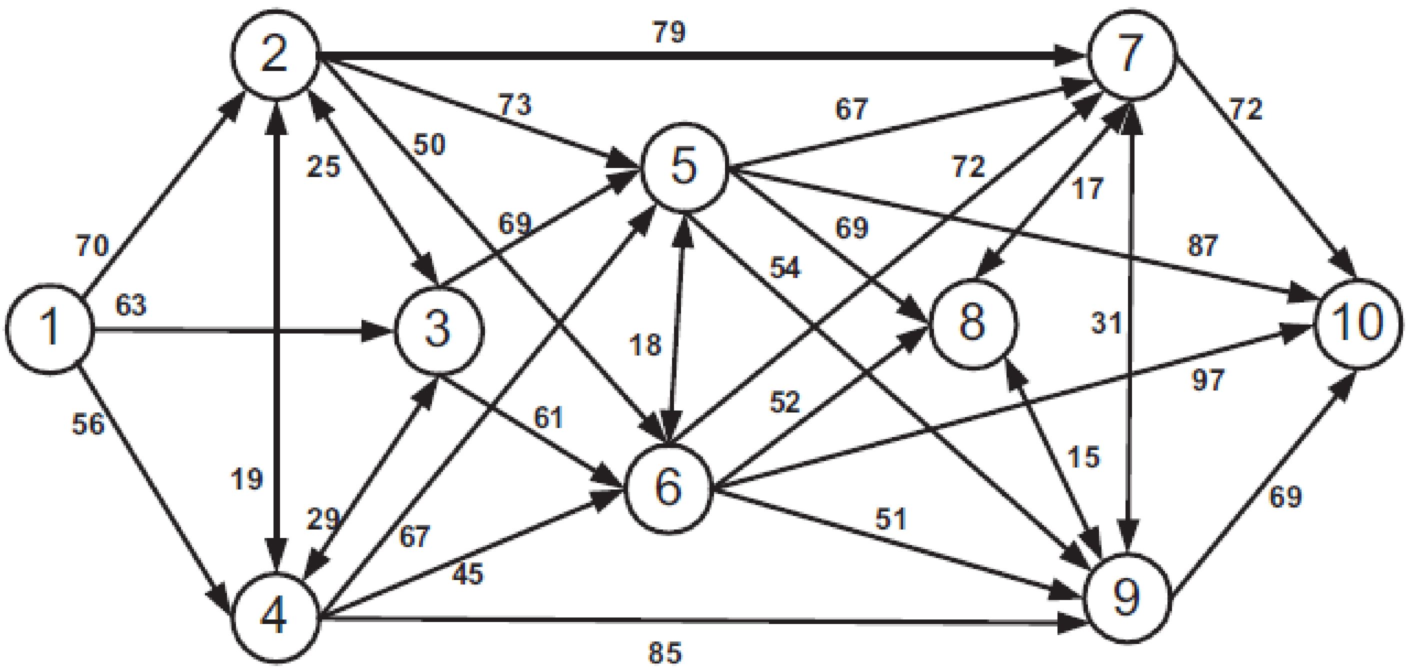


Example:
Network with flight times between city pairs



Solution

- Decision Variable: ?
- Objective function: ?
- Constraints: ?



Decision Variable

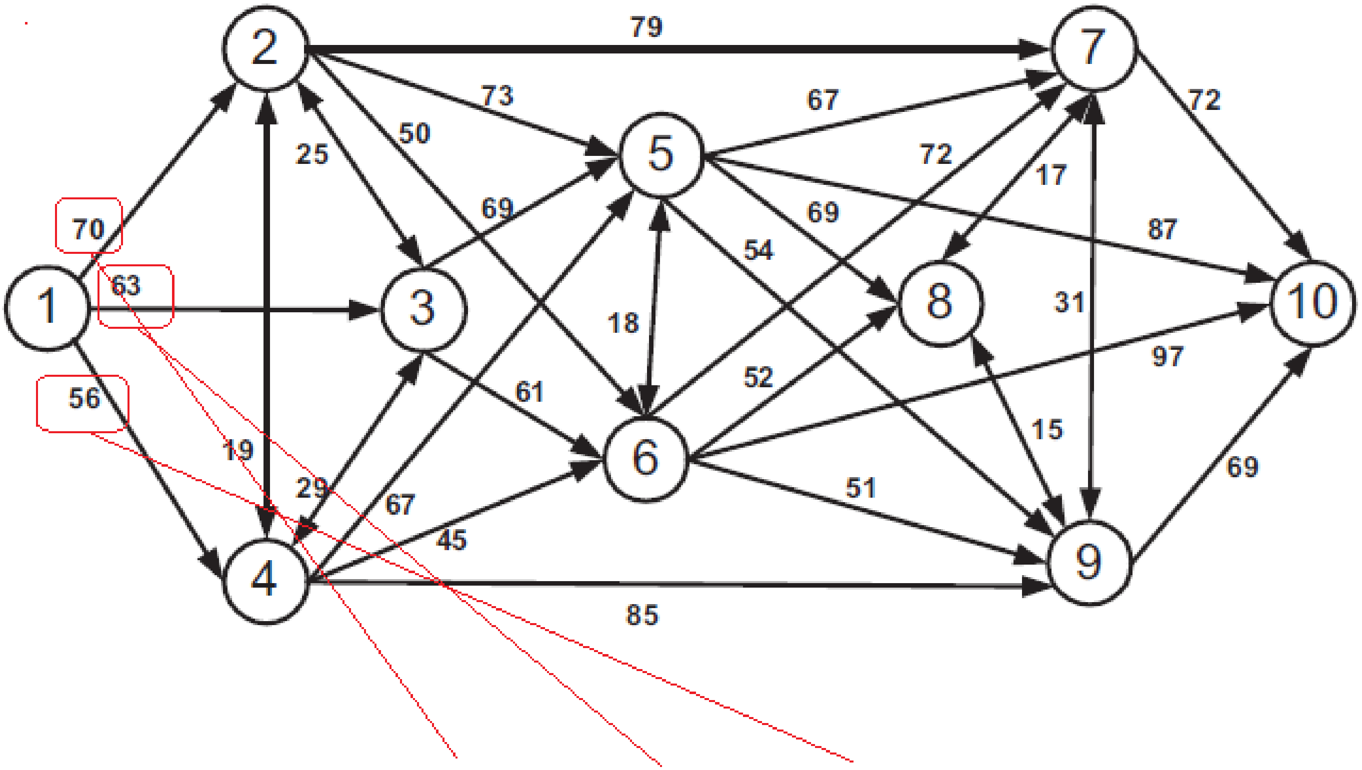
We assume the following binary (0-1) decision variable:

$$x_{i,j} = \begin{cases} 1 \text{ if arc } (i,j) \text{ is part of the solution} \\ 0 \text{ otherwise} \end{cases}$$

Objective Function

Minimise:

$$70x_{1,2} + 63x_{1,3} + 56x_{1,4} + \ldots + 72x_{7,10} + 87x_{5,10} + 97x_{6,10} + 69x_{9,10}$$

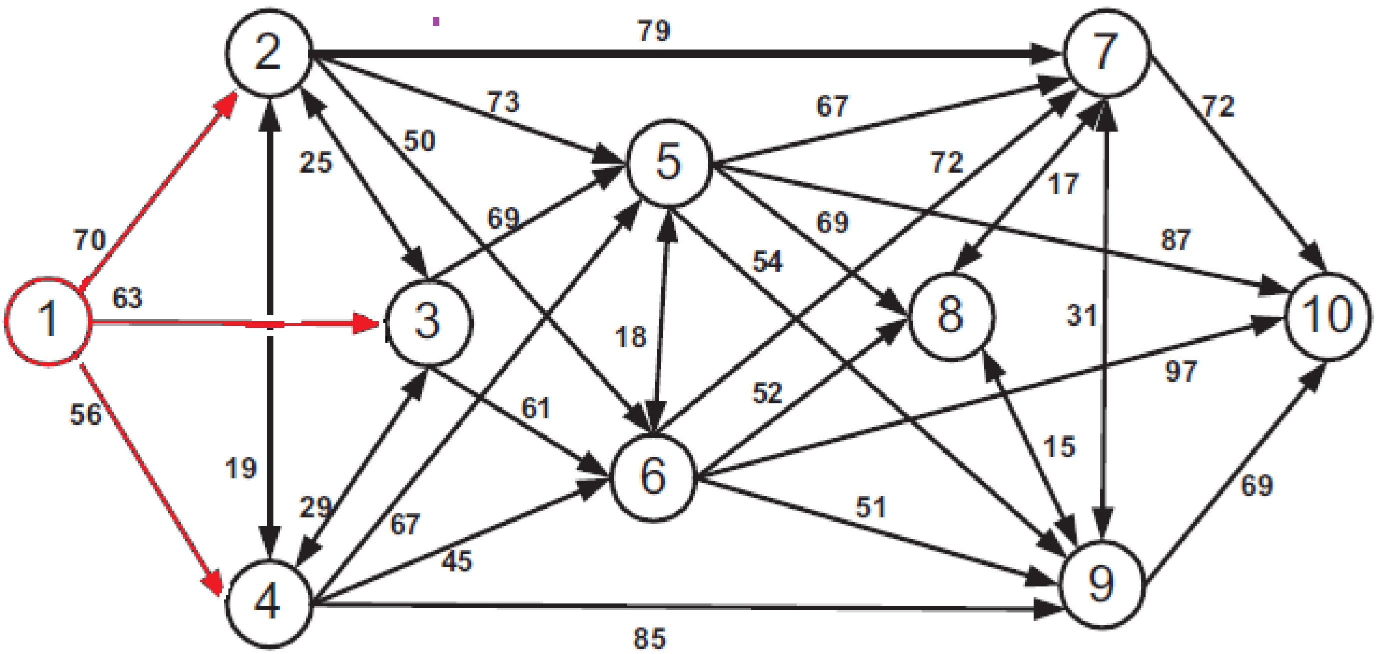


$$\textit{Minimize } 70x_{1,2} + 63x_{1,3} + 56x_{1,4} + \ldots..$$

Constraints

- Source node (Origin node): the flow must originate from node 1 and it must be a part of the solution, therefore:

$$x_{1,2} + x_{1,3} + x_{1,4} = 1$$



$$x_{1,2} + x_{1,3} + x_{1,4} = 1$$

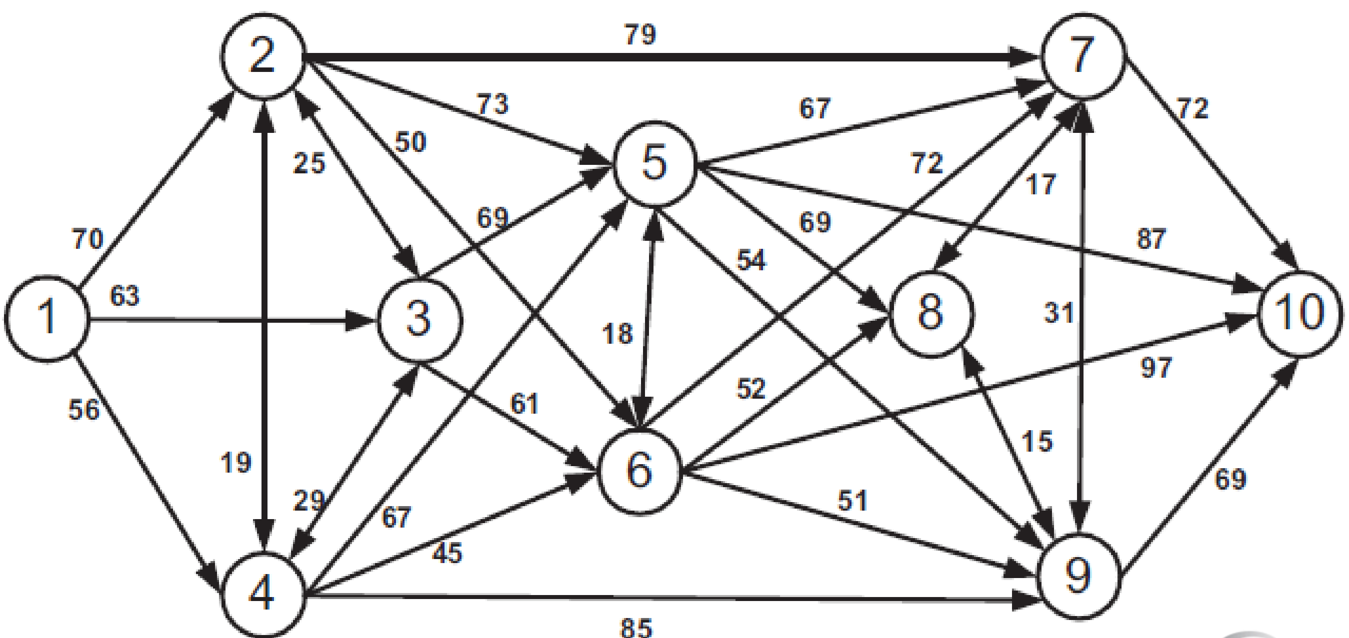
Constraints (Con't)

- Source node (Origin node):

$$x_{1,2} + x_{1,3} + x_{1,4} = 1$$

- Same as Destination node: the flow must end up at the destination node (node 10).

$$x_{5,10} + x_{6,10} + x_{7,10} + x_{9,10} = 1$$



Constraints for Transshipment Nodes

- Transshipment nodes: every other node (except origin and destination) is a transshipment node. The net flow in these nodes should be zero since good are not left there, they must be sent to the final destinations.
- The constraints for those transshipment nodes

$$x_{1,2} + x_{4,2} + x_{3,2} - x_{2,3} - x_{2,4} - x_{2,5} - x_{2,6} - x_{2,7} = 0$$

$$x_{1,3} + x_{2,3} + x_{4,3} - x_{3,2} - x_{3,4} - x_{3,5} - x_{3,6} = 0$$

$$x_{1,4} + x_{2,4} + x_{3,4} - x_{4,2} - x_{4,3} - x_{4,5} - x_{4,6} - x_{4,9} = 0$$

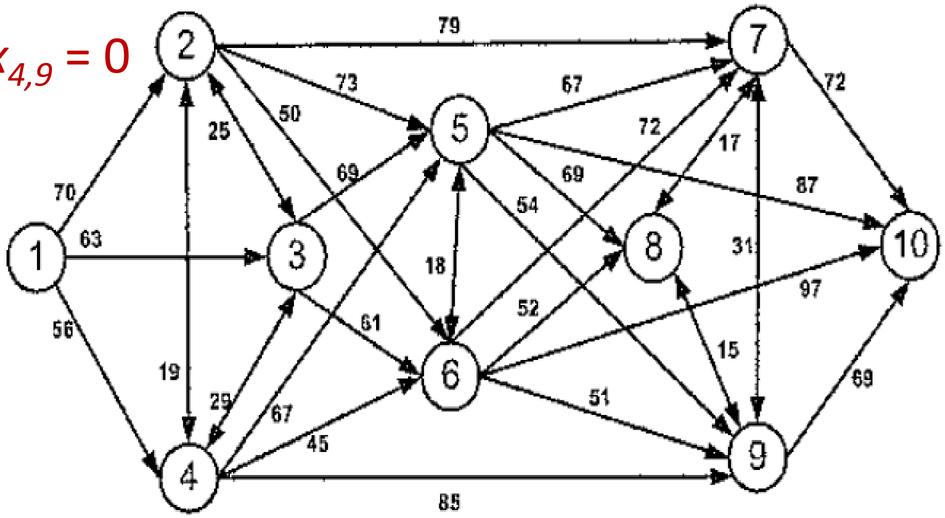
$$x_{2,5} + \ldots = 0$$

$$x_{2,6} + \ldots = 0$$

$$x_{2,7} + \ldots = 0$$

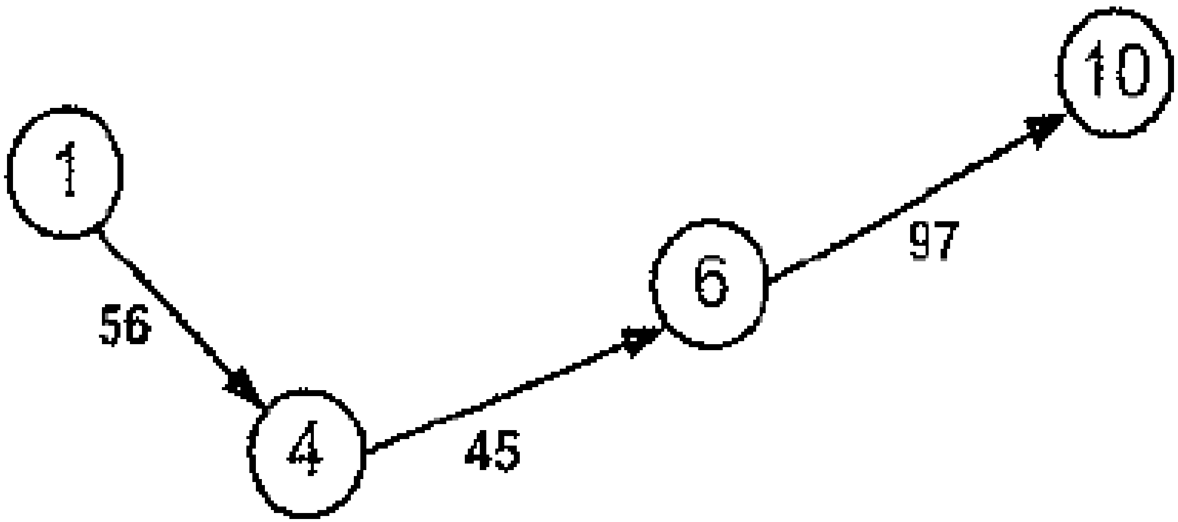
$$x_{5,8} + \ldots = 0$$

$$x_{4,9} + \ldots = 0$$



Final solution

- The minimum cost is 198
- The shortest path is 1-4-6-10



Summary

- Sets

  M = Set of nodes

- Index

  i, j, k = Index for nodes

- Parameters

  $C_{i,\,j}$ = Cost of flow along the arc joining node *i* to node *j*

  m = Destination node

Summary (Con't)

- Decision Variable

$$x_{i,j} = \begin{cases} 1 \text{ if arc } (i,j) \text{ is part of the path} \\ 0 \text{ otherwise} \end{cases}$$

- Objective Function

$$Minimize \sum_{i \in M} \sum_{j \in M} c_{i,j} \, x_{i,j}$$

- Subject to

$$\sum_{j \in M} x_{1,j} = 1 \quad j \neq 1$$

$$\sum_{j \in M} x_{i,j} - \sum_{k \in M} x_{k,i} = 0 \qquad For\ all\ (\forall)\ i,\ i \neq 1\ and\ i \neq m$$

$$\sum_{i \in M} x_{i,m} = 1$$

Network Flow models

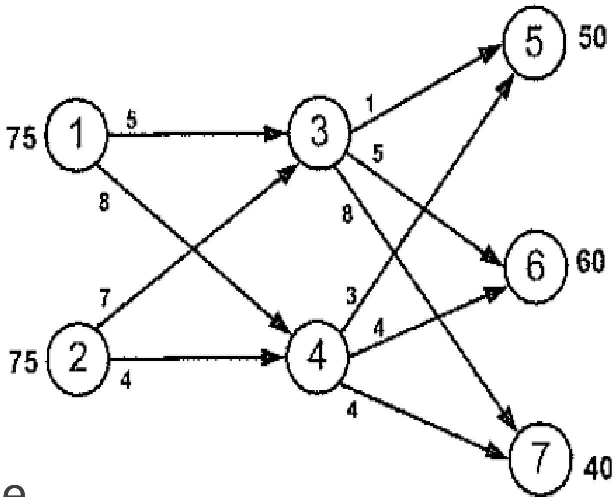
1. Shortest Path (Router) problem
2. Minimum cost flow problem
3. Maximum flow problem
4. Multi-commodity Problem

2. Minimum cost flow problem

- Seek to satisfy the requirements of nodes at minimum cost.
- It is a generalized form of transportation, transshipment, and shortest path problems.
- The problem assume the cost per unit of flow and capacities associated with each arc are known.

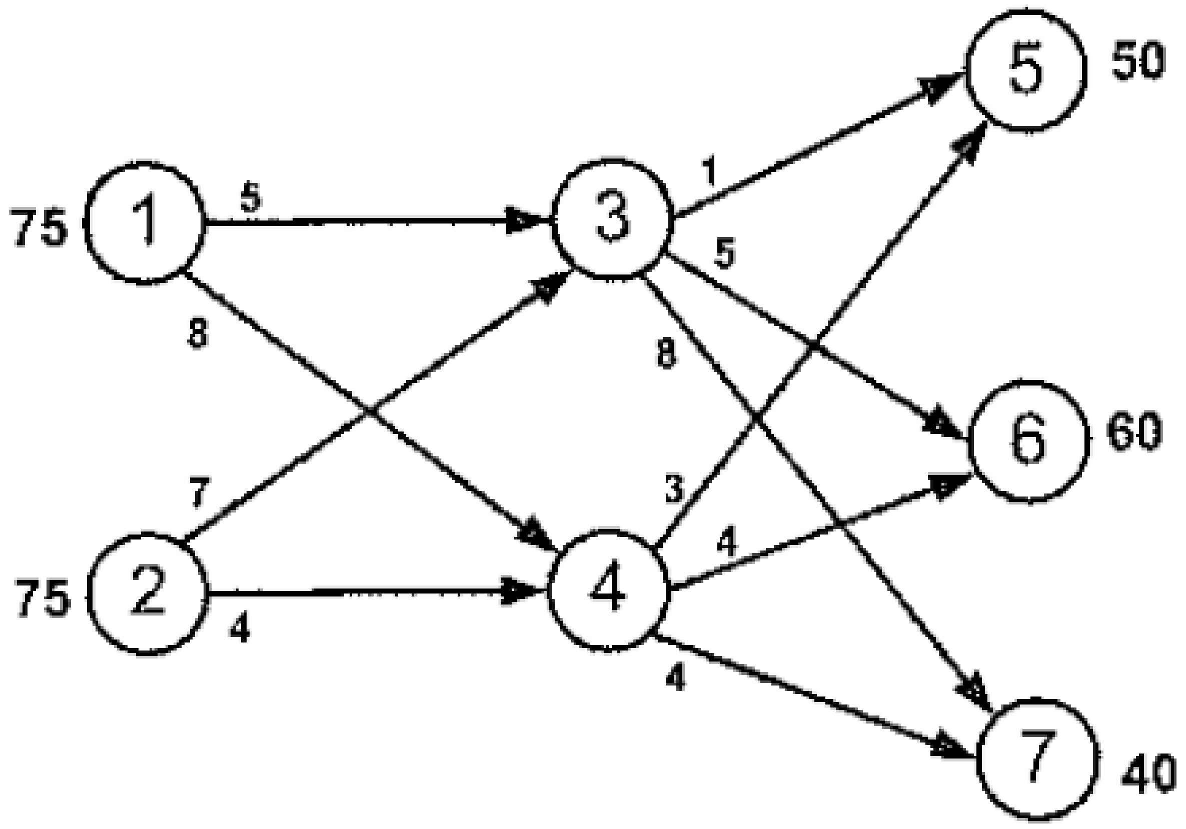
Air cargo problem description

- An airline is tasked with transporting goods from nodes 1 and 2 to nodes 5, 6, 7 in the diagram.
- The airline does not have direct flights from the source nodes to destination nodes, but they have the network which can connect those nodes through their hubs in nodes 3 & 4.
- The numbers next to the nodes represent the demands/supply in tons.
- Objective: determine the best way to transport the goods from sources to destinations to minimise the total cost.
- Constraint: The aircraft flying to and from node 4 can carry a maximum of 50 tons of cargo



Network presentation for minimum cost flow

- Note: If we carry 75 tons of cargo from node 1 to node 3, the cost is 5*75=375



Solution: To formulate this cargo problem

- Decision variable: $x_{i,j}$ = amount of flow from node i to node j

- The objective function is:

  Minimize: $5x_{1,3} + 8x_{1,4} + 7x_{2,3} + \dots$

- The constraints:

  $x_{1,3} + x_{1,4} \le 75$

  $x_{2,3} + x_{2,4} \le 75$

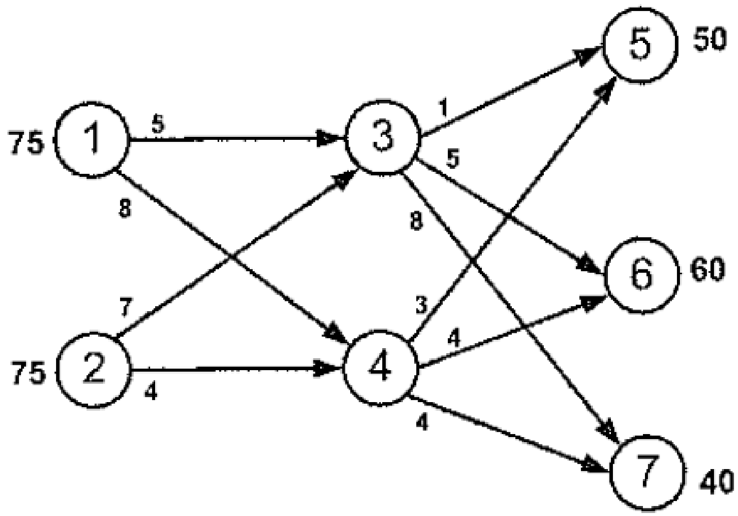
  $x_{1,3} + x_{2,3} - x_{3,5} - x_{3,6} - x_{3,7} = 0$

  $x_{1,4} + x_{2,4} - x_{4,5} - x_{4,6} - x_{4,7} = 0$

  $x_{3,5} + x_{4,5} = 50$

  $x_{3,6} + x_{4,6} = 60$

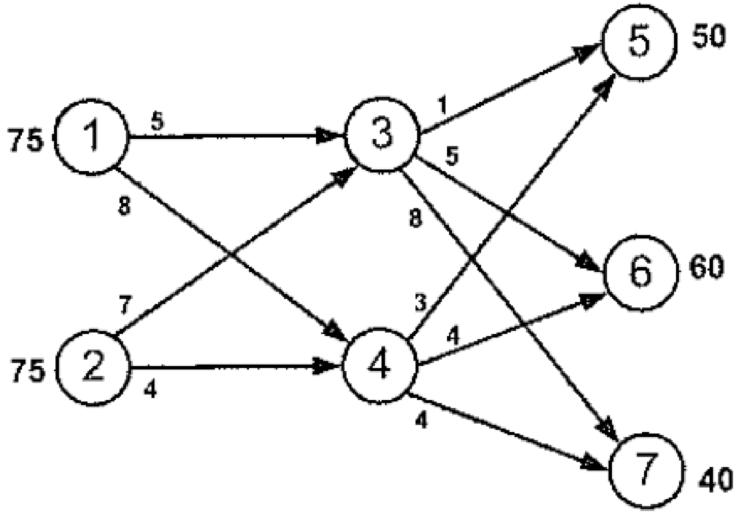
  $x_{3,7} + x_{4,7} = 40$



Additional constraints about capacity

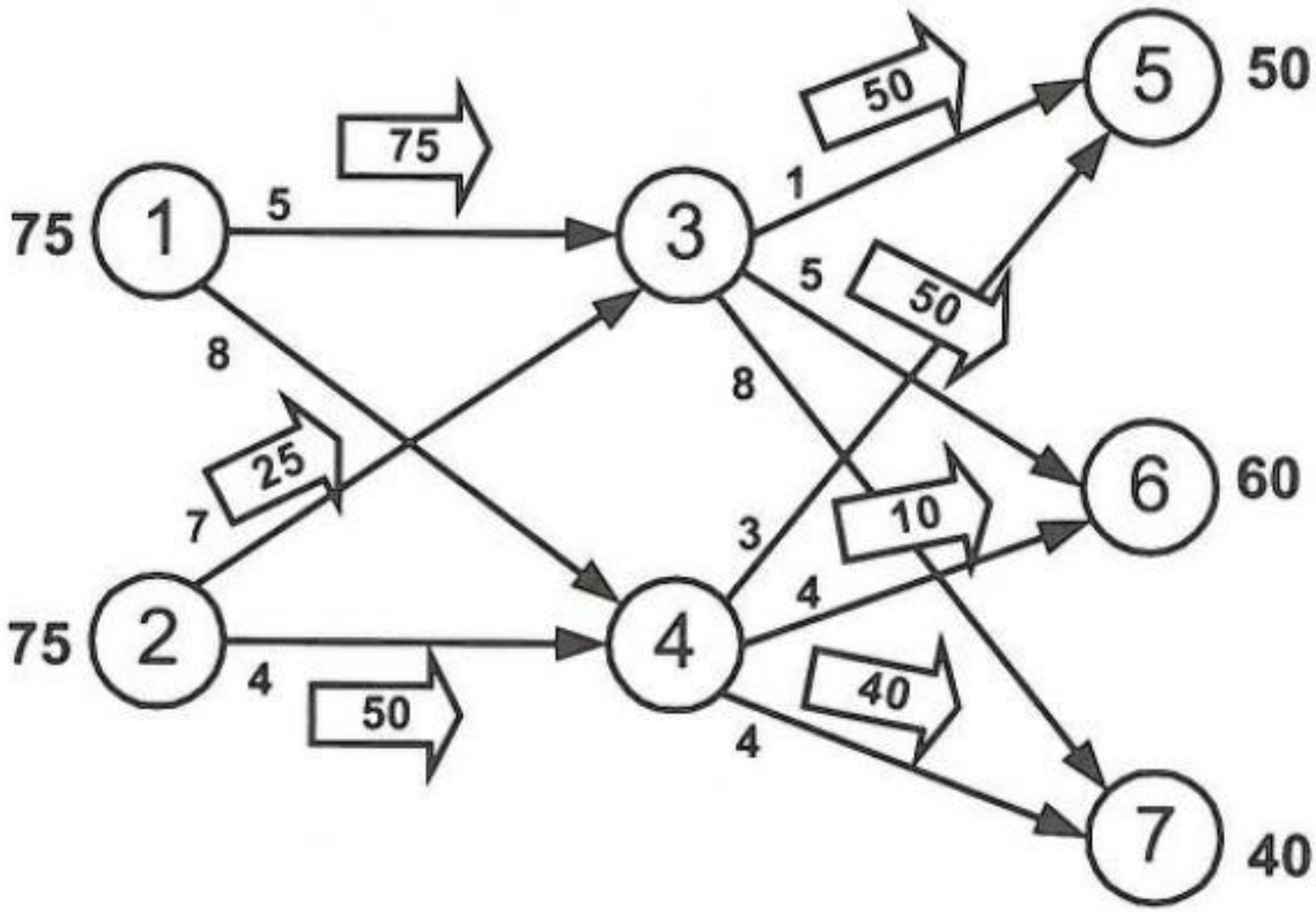
- All the flights to and from node 4 (Airport Y) can carry a maximum of 50 tons (might because of the shorter length of its runaway)

$$x_{1,4} \leq 50$$
$$x_{2,4} \leq 50$$
$$x_{4,5} \leq 50$$
$$x_{4,6} \leq 50$$
$$x_{4,7} \leq 50$$



Final Solution: Air cargo

- Total Minimum cost of $1,250.



Network Flow models

1. Shortest Path (Router) problem
2. Minimum cost flow problem
3. Maximum flow problem
4. Multi-commodity Problem

3. Maximum Flow Problem

- A Special Case of Minimum Cost Flow Problem

- E.g. An airline must determine the number of daily connecting flights that can be arranged between Daytona Beach (DAB), Florida, and Lafayette (LAF), Indiana.

- Connecting flights must stop in Atlanta (ATL), Georgia, and then make one more stop in either Chicago(ORD), Illinois, or Detroit (DTW), Michigan.

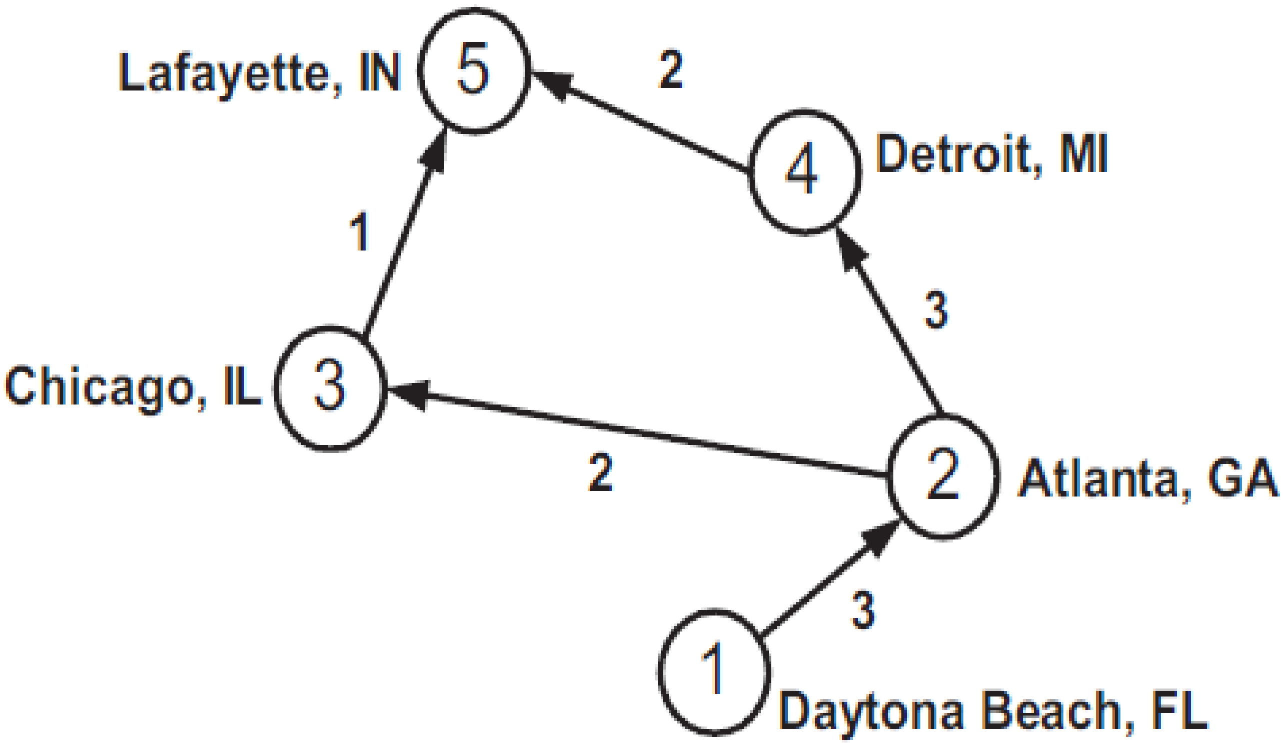
Example:
Maximum number of flights per city-pair

- The airline wants to determine how to maximize the number of connecting flights daily from Daytona Beach, FL, to Lafayette, IN, respecting the current restrictions.

City-Pairs	Maximum number of daily flights
DAB - ATL	3
ATL - ORD	2
ATL - DTW	3
ORD - LAF	1
DTW - LAF	2

Solution

- Work out the network presentation



Solution (Con't)

- **Decision Variable**

  Xij = Number of flights (integer from node i to node j)

  f = Number of daily flights from DAB to LAF

  - Maximize daily flights between DAB and LAF

  => Maximize f

- **Source Node**

  DAB is the source node, f is the total flow leaving DAB

  $x_{1,2} = f$

Solution (Con't)

- Transshipment nodes (example node 2 ATL)
  - $x_{1,2} - x_{2,3} - x_{2,4} = 0$
  - Similarly, transshipment constraints for other nodes 3 and 4.

- Destination node
  - same number of daily flights f departing from DAB should now arrive at destination node LAF.
  - $x_{4,5} + x_{3,5} = f$

Solution (Con't)

- Arc capacity

- The last set of constraints address the capacity of arcs as follows:

$$x_{1,2} \leq 3$$

$$x_{2,3} \leq 2$$

$$x_{2,4} \leq 3$$

$$x_{3,5} \leq 1$$

$$x_{4,5} \leq 2$$

Final Solution

- Solving this problem generates a maximum flow of three daily flights between DAB and LAF as follows:

  - 1 flight assigned to the DAB-ATL-ORD-LAF route, and;
  - 2 flights assigned to the DAB-ATL-DTW-LAF route.

Summary

Sets

M = Set of nodes

Index

i,j,k = Index for nodes

Parameters

$L_{i,j}$ = Lower bound on flow through arc (i,j)
$U_{i,j}$ = Upper bound on flow through arc (i,j)
m = Destination node

Decision Variables:

$x_{i,j}$ = Amount of flow from node i to node j
f = Amount of flow from source node to destination node

Summary (Con't)

$Maximize\ f$

$Subject\ to$

$$\sum_{j \in M} x_{1,j} = f \qquad \Leftrightarrow \text{ Origin Node}$$

$$\sum_{i \in M} x_{i,j} - \sum_{k \in M} x_{j,k} = 0 \qquad \Leftrightarrow \text{ Transshipment nodes}$$

$$\sum_{i \in M} x_{i,m} = f \qquad \Leftrightarrow \text{ Destination node}$$

$$L_{i,j} \le x_{i,j} \le U_{i,j}$$

Network Flow models

1. Shortest Path (Router) problem
2. Minimum cost flow problem
3. Maximum flow problem
4. Multi-commodity Problem

4. Multi-Commodity Problem

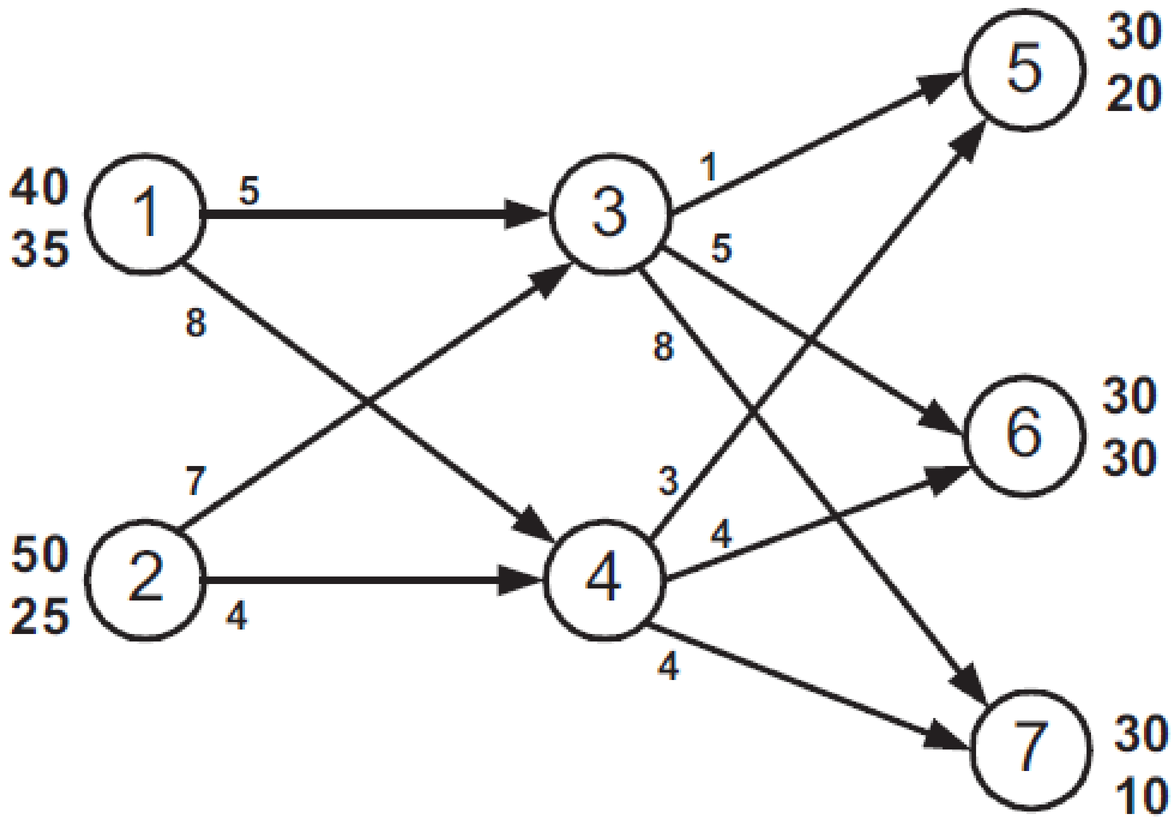
- All the network models explained so far assume that a single commodity or type of entity is sent through a network.

- A network may transport different types of commodities.

- The multi-commodity problem seeks to minimize the total cost when different types of goods are sent through the same network.

Example

- We modify the example that was presented for the Minimum Cost Flow problem discussed earlier to address the multi-commodity model formulation.



Solution

- **Decision Variable**
  - To formulate this problem we assume the following decision variable.
  - In this decision variable the indices i and j represent the nodes (i,j = 1,.,7) and k represents the type of commodity (k = 1,2).
  - x i,j,k = Amount of flow from node i to node j for commodity k
- **Objective Function**
  - Minimize $5x_{1,3,1} + 5_{x1,3,2} + 8x_{1,4,1} + 8x_{1,4,2} + \ldots$

Solution (Con't)

- We need to write one constraint for each node. For example, for node 1 we have:

$$x_{1,3,1} + x_{1,4,1} \leq 40$$

$$x_{1,3,2} + x_{1,4,2} \leq 35$$

- Similar constraint for the other six nodes

Solution (Con't)

- All the flights to and from node 4 can carry a maximum of 50 tons

$$x_{1,4,1} + x_{1,4,2} \leq 50$$

$$x_{2,4,1} + x_{2,4,2} \leq 50$$

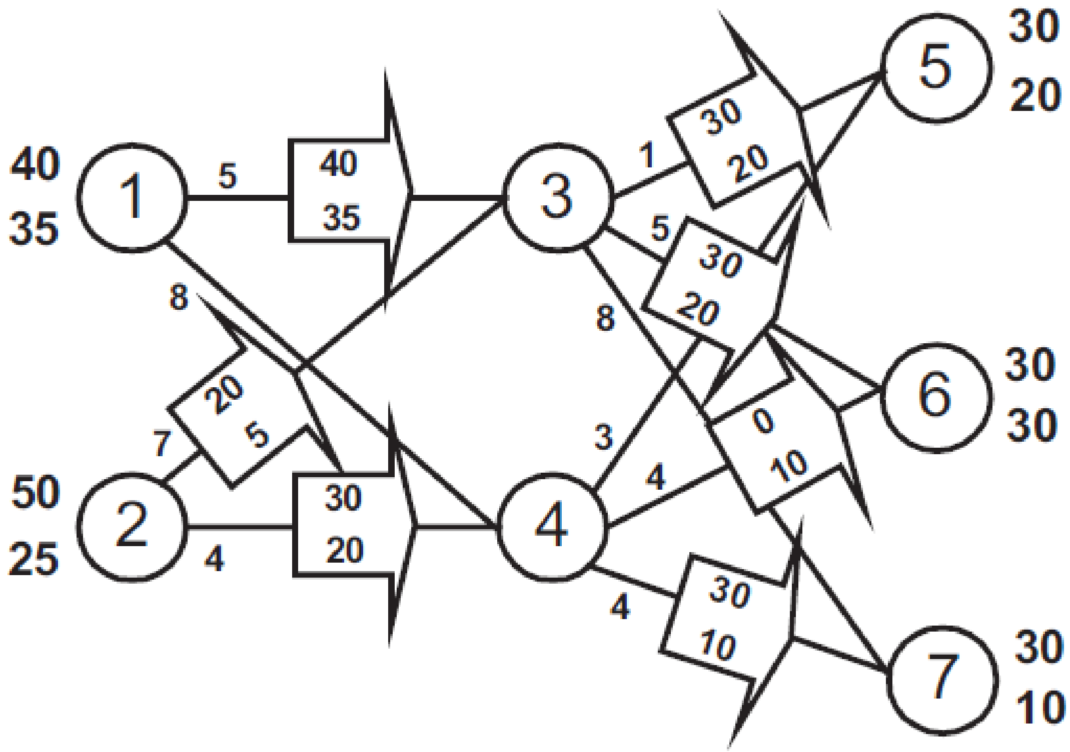
$$x_{4,5,1} + x_{4,5,2} \leq 50$$

$$x_{4,6,1} + x_{4,6,2} \leq 50$$

$$x_{4,7,1} + x_{4,7,2} \leq 50$$

Final Solution

- Solving this problem using software generates a total minimum cost of $1,250.



Key References

- M. Bazargan (2010) Airline Operations and Scheduling. 2nd edition, Ashgate
  - Chapter 2 Network Flows and Integer Programming Models